

ASYMPTOTIC BEHAVIOR OF TOR OVER COMPLETE INTERSECTIONS AND APPLICATIONS

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ABSTRACT. Let R be a local complete intersection and M, N are R -modules such that $\ell(\mathrm{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$. Imitating an approach by Avramov and Buchweitz, we investigate the asymptotic behavior of $\ell(\mathrm{Tor}_i^R(M, N))$ using Eisenbud operators and show that they have well-behaved growth. We define and study a function $\eta^R(M, N)$ which generalizes Serre's intersection multiplicity $\chi^R(M, N)$ over regular local rings and Hochster's function $\theta^R(M, N)$ over local hypersurfaces. We use good properties of $\eta^R(M, N)$ to obtain various results on complexities of Tor and Ext, vanishing of Tor, depth of tensor products, and dimensions of intersecting modules over local complete intersections.

1. INTRODUCTION

In this article we define and study a certain function on pairs of modules (M, N) over a local complete intersection R . We first recall some notions which inspire our work. In 1961, Serre defined a notion of intersection multiplicity for two finitely generated modules M, N over a regular local ring R with $\ell(M \otimes_R N) < \infty$ as:

$$\chi^R(M, N) = \sum_{i \geq 0} (-1)^i \ell(\mathrm{Tor}_i^R(M, N))$$

In 1980, Hochster defined a function $\theta^R(M, N)$ for a pair of finitely generated modules M, N over a local hypersurface R such that $\ell(\mathrm{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$ as:

$$\theta^R(M, N) = \ell(\mathrm{Tor}_{2e+2}^R(M, N)) - \ell(\mathrm{Tor}_{2e+1}^R(M, N)).$$

where e is an integer such that $e \geq d/2$. It is well known (see [Ei]) that $\mathrm{Tor}^R(M, N)$ is periodic of period at most 2 after $d+1$ spots, so this function is well-defined. The vanishing of this function over certain hypersurfaces was shown by Hochster to imply the Direct Summand Conjecture. This function is related to what is called ‘‘Herbrand difference’’ by Buchweitz in [Bu]. Hochster's theta function has recently been exploited in [Da1, Da2, Da4] to study a number of different questions on hypersurfaces, giving new results on rigidity of Tor, dimensions of intersecting cycles, depth of Hom and tensor products, splitting of vector bundles. These results extend works in [AG2, Au1, Au2, Fa, HW1, HW2, Jot, MNP] and provide some surprising new links between the classical homological questions that have been an active part of Commutative Algebra in the last 50 years.

Our main goal is to define, for a local complete intersection R , a function on any pair of R -modules (M, N) satisfying $\ell(\mathrm{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$ which can be viewed as a generalization of both Serre's intersection multiplicity and Hochster's theta function. Our definition will be asymptotic, since over complete intersections, free resolutions of modules do not have any obvious ‘‘finite’’ or ‘‘periodic’’ property.

Therefore, a study of the growth of lengths of the Tor modules is an essential first step.

Our approach for such task is parallel to that of a recent beautiful paper by Avramov and Buchweitz ([AB1]). The theory of complexity, which measures the polynomial growth of the Betti numbers of a module, has long been an active subject of Commutative Algebra. In their paper, Avramov and Buchweitz studied complexity for Ext modules using Quillen's approach to cohomology of finite groups and the structure of the total module $\text{Ext}^*(M, N)$ as a Noetherian module over the ring of cohomology operators (in the sense of Eisenbud, see [Ei]). One of the technical difficulties for our approach is that over the ring of cohomology operators, it is not clear what structure $\text{Tor}_*(M, N)$ has. It is well known that if $\ell(M \otimes_R N) < \infty$ then $\text{Tor}_*(M, N)$ becomes an Artinian module, however our condition on lengths of Tor modules is weaker.

We begin, in the second section, by introducing a notion that is suitable to describing the structure of $\text{Tor}_*(M, N)$. Let $T = \bigoplus_i T_i$ be a \mathbb{N} -graded module over $S = R[x_1, \dots, x_r]$ such that the x_i s act with equal negative degree. T is called *almost Artinian* if there is an integer j such that $T_{\geq j} = \bigoplus_{i \geq j} T_i$ is Artinian over S . We collect basic properties for modules in this category. We also introduce a number of different notions of complexities for pairs of modules using Tor.

Section 3 is devoted to an investigation of the structure of the module $\text{Tor}_*(M, N)$ over the ring of homology operators. When $R = Q/(f_1, \dots, f_r)$ (here Q is not necessarily regular), these operators are R -linear maps $x_j : \text{Tor}_{i+2}^R(M, N) \rightarrow \text{Tor}_i^R(M, N)$ for $1 \leq j \leq r$ and $i \geq 0$. We show that when $\ell(\text{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$, the module $\text{Tor}_*^R(M, N)$ is almost Artinian over the ring of homology operators $S = R[x_1, \dots, x_r]$. We also show that when $\ell(\text{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$, $\text{Tor}_*^R(M, N)$ is almost Artinian over S if and only if $\text{Tor}_*^Q(M, N)$ is almost Artinian over Q , that is, $\text{Tor}_i^Q(M, N) = 0$ when $i \gg 0$. This is an analogue of results obtained by Gulliksen, Avramov-Gasharov-Peeva and is crucial in our analysis of lengths of Tor modules later. Our proof is rather *ad hoc*, since almost Artinian modules do not behave as well as Artinian or Noetherian ones, (see Example 2.9).

In section 4 we study the “adjusted lengths” (which is equal to normal length, except when the module does not have finite length, then it is equal to 0) of $\text{Tor}_i^R(M, N)$, which we call *generalized Betti numbers* $\beta_i(M, N)$. We prove properties for these numbers which subsume previously known results about Betti numbers of a module over local complete intersections. We arrive at the main goal of this note, the definition of a function:

$$\eta_e^R(M, N) := \lim_{n \rightarrow \infty} \frac{\sum_0^n (-1)^i \beta_i(M, N)}{n^e}$$

We show that $\eta_e^R(M, N)$ is finite when e is at least the “Tor-complexity” of (M, N) , and it is additive on short exact sequences provided that it is defined on all pairs involved. We also obtain a change of rings result which relates the values of $\eta^R(M, N)$ and $\eta^{R/(f)}(M, N)$ where f is a nonzerodivisor on R .

In section 5, we compare the several notions of complexities that arise in our work with the one previously studied by Avramov and Buchweitz. We prove they coincide when all the higher Tor modules have finite length, generalizing (see 5.5) a striking result in [AB1] that over a complete intersection, the vanishing of all higher Tor modules is equivalent to the vanishing of all higher Ext modules. The

connections between the complexities in general seems like a difficult problem and are worth further investigation.

The rest of the paper is concerned with applications. In section 6 we study vanishing behavior of $\text{Tor}_i^R(M, N)$. The key idea is to use our results about rigidity over hypersurface in [Da1] as the base case and our change of rings theorem for $\eta^R(M, N)$ for the inductive step. Our results in this situation improve on results in [Mu, Jo1, Jo2]. In section 7, the main results basically say that under some extra conditions, good depth of $M \otimes_R N$ forces the vanishing of $\text{Tor}_i^R(M, N)$ for all $i > 0$. These are generalizations of Auslander's classical result on tensor product over regular local rings and in some senses improve upon similar results in complete intersections of small codimensions by Huneke, Jorgensen, and Wiegman in [HJW]. Finally, in section 8 we give some applications on intersection theory over local complete intersections, extending results by Hochster and Roberts in [Ho1, Ro]. We also discuss some interesting questions that give new perspectives on a classical conjecture that grew out of Serre's work on intersection multiplicity. We remark that the above list contains only the most obvious applications of $\eta^R(M, N)$. More technical ones, for example generalizations of results by Auslander and Auslander-Goldman, will be the topics of forthcoming papers.

The main ideas in this article can almost certainly be applied to study vanishing of Ext modules over complete intersections. In fact, the module structure of $\text{Ext}_R^*(M, N)$ over the ring of cohomology operators is much better understood (see [AB1, AB2, AGP]). We focus our study on Tor , since many of the open homological questions could be viewed as problems about length of Tor modules.

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2. NOTATION AND PRELIMINARY RESULTS

Throughout this section, let (R, m, k) be a Noetherian local ring. Let M, N be finitely generated R -modules. Let $S = R[x_1, \dots, x_r]$ for indeterminates x_1, \dots, x_r and $T = \bigoplus_{i \geq 0} T_i$ be a \mathbb{N} -graded module over S . We shall start by making or recalling some definitions.

We define the *adjusted length* of M as :

$$al_R(M) := \begin{cases} \ell_R(M) & \text{if } \dim(M) = 0 \\ 0 & \text{if } \dim(M) > 0 \end{cases}$$

The minimal number of generators of M is denoted by $\mu(M)$.

We define the *finite length index* of T as :

$$f_R(T) := \inf \{i \mid \ell_R(T_j) < \infty \text{ for } j \geq i\}$$

(If no such i exists, we set $f_R(T) = \infty$).

The *complexity* of a sequence of integers $B = \{b_i\}_{i \geq 0}$ is defined as:

$$\text{cx}(B) := \inf \{d \in \mathbb{Z} \mid b_n \leq an^{d-1} \text{ for some real number } a \text{ and all } n \gg 0\}$$

If $f_R(T) < \infty$ then one can define the complexity of T as:

$$\text{cx}_R(T) := \text{cx}(\{al_R(T_i)\})$$

For a pair of R -modules M, N , let $\mathrm{Tor}_*^R(M, N) = \bigoplus_i \mathrm{Tor}_i^R(M, N)$ and $\mathrm{Ext}_R^*(M, N) = \bigoplus_i \mathrm{Ext}_R^i(M, N)$. The concept of *complexity* for (M, N) was first introduced in [AB1]. In our notations, their definition becomes:

$$\mathrm{cx}_R(M, N) := \mathrm{cx}_R(\mathrm{Ext}_R^*(M, N) \otimes_R k)$$

Similarly we will define several analogues of $\mathrm{cx}_R(M, N)$. The *Tor complexity* of M, N is:

$$\mathrm{tcx}_R(M, N) = \mathrm{cx}_R(\mathrm{Tor}_*^R(M, N) \otimes_R k)$$

The *length complexity* of M, N is:

$$\mathrm{lcx}_R(M, N) = \mathrm{cx}_R(\mathrm{Tor}_*^R(M, N))$$

Definition 2.1. Let $T = \bigoplus_{i \geq 0} T_i$ be a \mathbb{N} -graded module over $S = R[x_1, \dots, x_r]$ such that the x_i s act with equal negative degree. T is called *almost Artinian* over S if there is an integer j such that $T_{\geq j} = \bigoplus_{i \geq j} T_i$ is Artinian over S .

Now we collect some results that will be needed. Some were already in the literature, but for some we can not find a reference. First, we recall the long exact sequence for change of rings (2.2). It follows from the Cartan-Eilenberg spectral sequence ([Av2], 3.3.2).

Proposition 2.2. Let $R = Q/(f)$ such that f is a nonzerodivisor on Q , and let M, N be R -modules. Then we have the long exact sequence of Tors :

$$\begin{aligned} \dots &\rightarrow \mathrm{Tor}_n^R(M, N) \rightarrow \mathrm{Tor}_{n+1}^Q(M, N) \rightarrow \mathrm{Tor}_{n+1}^R(M, N) \\ &\rightarrow \mathrm{Tor}_{n-1}^R(M, N) \rightarrow \mathrm{Tor}_n^Q(M, N) \rightarrow \mathrm{Tor}_n^R(M, N) \\ &\rightarrow \dots \\ &\rightarrow \mathrm{Tor}_0^R(M, N) \rightarrow \mathrm{Tor}_1^Q(M, N) \rightarrow \mathrm{Tor}_1^R(M, N) \rightarrow 0 \end{aligned}$$

Theorem 2.3. ([Ki], theorem 1) Suppose that all the x_i have equal negative degree. Then T is an Artinian S module if and only if there are integers n, l such that :

- (1) $T_i = 0$ for $i < l$
- (2) $0 :_{T_i} (\sum_{j=1}^r x_j R) = 0$ for $i > n$.
- (3) T_i is an Artinian R -module for all i .

Now suppose that the x_i have equal positive degree d . Then T is an Noetherian S module if and only in there are integer n, l such that:

- (4) $T_i = 0$ for $i < l$
- (5) $T_{i+d} = \sum_{j=1}^r x_j T_i$ for all $i > n$.
- (6) T_i is an Noetherian R -module for all i .

Corollary 2.4. Assume that the x_i have equal positive degree d and T is Noetherian over S . Then the sequence of ideals $\{\mathrm{ann}_R(T_i)\}$ eventually becomes periodic of period d .

Proof. By 2.3 you can choose n such that $T_{i+d} = \sum_{j=1}^r x_j T_i$ for all $i > n$. Let $J_i = \mathrm{ann}_R(T_i)$. Then $J_i \subseteq J_{i+d}$ for $i > n$. Since R satisfies ACC, the conclusion follows. \square

Corollary 2.5. Let T, T' be such that $\ell(T_i), \ell(T'_i) < \infty$ for all i and $T_i = T'_{i+e}$ for some e and all $i \gg 0$. Then T is almost Artinian if and only if T is Artinian if and only if T' is Artinian.

Lemma 2.6. *Let $0 \rightarrow T' \rightarrow T \rightarrow T'' \rightarrow 0$ be a short exact sequence of S modules such as the maps are homogenous. Then T is almost Artinian if and only if T', T'' are almost Artinian.*

Proof. This is obvious from the definitions. □

Lemma 2.7. ([Ei], 3.3) *Let $R = k$ be a field and assume k is infinite. Also assume that all the x_i are of equal negative degree $-d$, and T is an Artinian S module. Then there are elements $\alpha_i \in k$ such that multiplication by $x = x_r + \sum_{i=2}^{r-1} \alpha_i x_i$ induces a surjective map $T_{i+d} \rightarrow T_i$ for all $i \gg 0$.*

Lemma 2.8. ([Gu], 1.3) *Let $x : T \rightarrow T$ be a homogenous S -linear map of negative degree. Assume that $\ker(x)$ is an Artinian S -module. Then T is an Artinian $S[x]$ -module.*

Example 2.9. Gulliksen result above fails for almost Artinian module. Let R be a local ring such that $\dim R > 0$. Take T such that $T_i = R$, $x : T_{i+1} \rightarrow T_i$ be the identity map for all i and $S = R$. Then $\ker(x) = T_0$ is clearly almost Artinian. However, T is not an almost Artinian $S[x]$ -module, since this would imply, by definition and 2.3, that each T_i is Artinian for $i \gg 0$.

Lemma 2.10. *Let I be an R -ideal that kills M . Assume I is m -primary. Then:*

$$\mu(M)\ell(R/I) \geq \ell(M) \geq \mu(M)$$

Proof. The left inequality follows by tensoring the surjection:

$$R^{\mu(M)} \rightarrow M \rightarrow 0$$

with R/I to get :

$$(R/I)^{\mu(M)} \rightarrow M \rightarrow 0$$

The right inequality follows from the surjection:

$$M \rightarrow M/mM \rightarrow 0$$

□

Lemma 2.11. *Let $T = \bigoplus_{i \geq 0} T_i$ be a graded $S = R[x_1, \dots, x_r]$ -module, with all x_i having equal negative degrees. Suppose $f_R(T) = 0$, (i.e that all T_i have finite length). Let T^\vee denote $\bigoplus_i \text{Hom}_R(T_i, E_R(k)) = \text{Hom}_{\text{graded } R\text{-modules}}(T, E_R(k))$, the graded Matlis dual. Then T^\vee becomes a graded module over $S' = R[x_1^\vee, \dots, x_r^\vee]$. Furthermore, T is an Artinian S -module if and only if T^\vee is a Noetherian S' -module.*

Proof. The first statement follows from the fact that Matlis dual is a contravariant functor. For the second statement, we just note that in this situation, being Noetherian (respectively Artinian) is the same as satisfying the ACC (respectively DCC) condition on graded submodules (one could use 2.3 to see this). Since $(-)^\vee$ gives a order-reserving bijection between the sets of graded submodules of T and T^\vee , we are done. □

Lemma 2.12. *Suppose that T is almost Artinian over S , with x_i of negative degree $-d_i$. Then one can write :*

$$P_T(t) = \sum_i a\ell(T_i)t^i = \frac{p(t)}{\prod_1^r (1 - t_i^{d_i})}$$

with $p(t) \in \mathbb{Z}[t]$.

Proof. Let $N = f_R(T)$. Then the graded module $T' = T_{\geq N}$ is an Artinian module over S . So we have:

$$P_T(t) = P_{T'}(t) + q(t)$$

for some $q(t) \in \mathbb{Z}[t]$. The result now follows from standard facts on Artinian modules over graded rings. \square

3. ALMOST ARTINIAN STRUCTURE OF $\text{Tor}_*^R(M, N)$

Let R, Q be local rings such that $R = Q/(f_1, \dots, f_r)$ and (f_1, \dots, f_r) is a regular sequence on Q . Let M, N be finitely generated R -modules. It is well known that $\text{Tor}_*^R(M, N)$ has a module structure over the ring of cohomology operators $S = R[x_1, \dots, x_r]$. We begin by reviewing standard facts about these operators induced by f_1, \dots, f_r .

Proposition 3.1. *Let R, Q, M, N be as above. Then for $1 \leq j \leq r$ there are R -linear maps:*

$$x_j : \text{Tor}_{i+2}^R(M, N) \rightarrow \text{Tor}_i^R(M, N)$$

for $i \geq 0$ which satisfy the following properties:

- (1) $\text{Tor}_*^R(M, N)$ becomes a graded module over the ring $S = R[x_1, \dots, x_r]$ (with each x_i has degree -2).
- (2) When $r = 1$, the map x_1 is (up to sign) the connecting homomorphism in the long exact sequence of Tors for change of rings 2.2.
- (3) For any $1 \leq s \leq r$, the operators x_1, \dots, x_s act on $\text{Tor}_*^R(M, N)$ in two ways: the initial one, from $R = Q/(f_1, \dots, f_r)$, and a new one, from the presentation $R = R'/(f_1, \dots, f_s)$ with $R' = Q/(f_{s+1}, \dots, f_r)$.

Proof. Part (1) is [Ei], Proposition 1.6. Part (2) and (3) follows from dualizing Proposition 2.3 in [Av1], which gave the corresponding result for Ext. Specifically, one should tensor the exact sequence (2.5.1) there with N . One can also deduce (2) and (3) from the detailed comparison of different constructions of cohomology operators done in [AS], Section 4. \square

In the following lemma we shall write $T^R(M, N)$ or sometimes T^R for $\text{Tor}_*^R(M, N)$.

Lemma 3.2. *Let R, Q, S, M, N as in Proposition 3.1. Assume that $f_R(T^R(M, N)) < \infty$. Then $T^R(M, N)$ is almost Artinian over S if and only if $T^Q(M, N)$ is almost Artinian over Q (which is equivalent to $\text{Tor}_i^Q(M, N) = 0$ for $i \gg 0$).*

Proof. Assume T^R is almost Artinian over S . Let $R' = Q/(f_1, \dots, f_{r-1})$ and consider the long exact sequence :

$$\longrightarrow T_{i+2}^R \xrightarrow{d_{i+2}} T_i^R \longrightarrow T_{i+1}^{R'} \longrightarrow T_{i+1}^R \xrightarrow{d_{i+1}} \longrightarrow$$

Note that the connecting maps d_i s are just multiplication by x_r (up to sign) in T^R . Break the long exact sequence into short exact sequences and assemble them together to get an exact sequence:

$$0 \rightarrow T^R/x_r T^R \rightarrow T^{R'} \rightarrow \ker(x_r) \rightarrow 0$$

The result follows from lemma 2.6 and induction on r (the case $r = 0$ is vacuous).

Now we prove the “if” part. Here a more careful argument is needed. This is because lemma 2.8 fails badly for almost Artinian modules (see 2.9). We still use

induction. By assumption there is an integer n such that $T_{\geq n}$ is Artinian. Truncate the long exact sequence for Tor above at T_{n+1}^R . It is obvious that $\ell(T_i^{R'}) < \infty$ for $i \geq n+1$. By the induction hypotheses, $T^{R'}$ is almost Artinian, hence so is $T_{i \geq n+1}^{R'}$. Consider the module $K = \oplus_{i \geq n+2} \ker(d_i)$. Then K is a graded quotient of $T_{i \geq n+1}^{R'}$, therefore it is Artinian over S . Restricting to $T_{i \geq n}^R$, $\ker(x_r) = K \oplus T_{n+1}^R \oplus T_n^R$. Since the last two summands are Artinian R -modules and K is an Artinian S -module, $\ker(x_r)$ is Artinian S -module. It follows from lemma 2.8 that $T_{i \geq n}^R$ is Artinian S -module, finishing our proof. \square

Corollary 3.3. *If $\ell(M \otimes_R N) < \infty$, then $T^R(M, N)$ is Artinian over S if and only if $T^Q(M, N)$ is Artinian over Q (if and only if $\text{Tor}_i^Q(M, N) = 0$ for $i \gg 0$).*

4. ASYMPTOTIC BEHAVIOR OF LENGTHS OF $\text{Tor}^R(M, N)$ AND THE FUNCTION $\eta^R(M, N)$

In this section we shall study the behavior of the lcx function (see Section 2) over complete intersections. We will only be concerned with R -modules M, N such that $f_R(M, N) = f_R(\text{Tor}_*^R(M, N)) < \infty$, that is, $\ell(\text{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$. For such pairs of modules, we define the *generalized Betti numbers* as :

$$\beta_i^R(M, N) = \ell(\text{Tor}_i^R(M, N))$$

Obviously, when $N = k$ we have the usual Betti numbers for M . The following result shows that well-known properties for Betti numbers still hold(cf. [Av1], 9.2.1):

Theorem 4.1. *Let R, Q be local rings such that $R = Q/(f_1, \dots, f_r)$ and f_1, \dots, f_r is a regular sequence on Q . Let M, N be R modules such that $\text{Tor}_i^Q(M, N) = 0$ for $i \gg 0$ (which is automatic if Q is regular) and $\ell(\text{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$. Let:*

$$P_{M,N}^R(t) = \sum_{i=0}^{\infty} \beta_i^R(M, N) t^i$$

We have:

- (1) *There is a polynomial $p(t) \in \mathbb{Z}[t]$ with $p(\pm 1) \neq 0$, such that*

$$P_{M,N}^R(t) = \frac{p(t)}{(1-t)^c(1+t)^d}$$

- (2) *For $i \gg 0$, there are equalities:*

$$\beta_i(M, N) = \frac{m_0}{(c-1)!} i^{c-1} + (-1)^i \frac{n_0}{(d-1)!} i^{d-1} + g_{(-1)^i}(i)$$

with $m_0 \geq 0$ and polynomials $g_{\pm}(t) \in \mathbb{Q}[t]$ of degrees $< \max\{c, d\} - 1$.

- (3) $d \leq c = \text{lcx}_R(M, N) \leq r$.

Proof. (1) This is due to the fact that $\text{Tor}_*^R(M, N)$ is an almost Artinian graded $R[x_1, \dots, x_r]$ -module, with the x_i s having degree -2 , and 2.12.

- (2) From part (1), we can write :

$$\sum_i \beta_i t^i = \frac{p(t)}{(1-t)^c(1+t)^d} = \sum_{l=0}^{c-1} \frac{m_l}{(1-t)^{c-l}} + \sum_{l=0}^{d-1} \frac{n_l}{(1+t)^{d-l}} + q(t)$$

Here $q(t) \in \mathbb{Z}[t]$. Then by comparing coefficients from both sides we get the desired formula for β_i . Since $\beta_i \geq 0$, m_0 must be greater than or equal to 0.

- (3) That $c \leq r$ is obvious. Since the sign of β_i for odd i is positive only if $c \geq d$, the first inequality is also clear. The size of β_i behaves like a polynomial of degree $\max\{c, d\} - 1 = c - 1$, which shows that $\text{lcx}_R(M, N) = c$. \square

The behavior of $\beta_i(M, N)$ allows us to define an asymptotic function on M, N as follows:

Definition 4.2. For a pair of R -modules M, N such that $f_R(M, N) < \infty$, and an integer $e \geq \text{lcx}_R(M, N)$, let :

$$\eta_e^R(M, N) := \lim_{n \rightarrow \infty} \frac{\sum_{i=0}^n (-1)^i \beta_i(M, N)}{n^e}$$

The following result shows that our η function behaves quite well:

Theorem 4.3. Let R, Q be local rings such that $R = Q/(f_1, \dots, f_r)$ and f_1, \dots, f_r is a regular sequence on Q . Let M, N be R modules such that $\text{Tor}_i^Q(M, N) = 0$ for $i \gg 0$ and $\ell(\text{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$. Let $c = \text{lcx}_R(M, N)$ and $e \geq c$ be an integer.

- (1) $\eta_e^R(M, N)$ is finite, and if $e > c$, $\eta_e^R(M, N) = 0$
- (2) (Biadditivity) Suppose that $e \geq 1$. Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ and $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ be exact sequences and suppose that for each i , $f_R(M_i, N)$ and $f_R(M, N_i)$ are finite. If $e \geq \max_i \{\text{lcx}(M_i, N)\}$ then:

$$\eta_e^R(M_2, N) = \eta_e^R(M_1, N) + \eta_e^R(M_3, N)$$

and if $e \geq \max_i \{\text{lcx}(M, N_i)\}$, then:

$$\eta_e^R(M, N_2) = \eta_e^R(M, N_1) + \eta_e^R(M, N_3)$$

If $\ell(M \otimes_R N) < \infty$, the conclusion also holds for $e = 0$.

- (3) (Change of rings) Let $r \geq 1$ and $R' = Q/(f_1, \dots, f_{r-1})$. Note that we also have $\ell(\text{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$. Assume that $e \geq 2$ and $e - 1 \geq \text{lcx}_{R'}(M, N)$. Then we have :

$$\eta_e^R(M, N) = \frac{1}{2e} \eta_{e-1}^{R'}(M, N)$$

If $\ell(M \otimes_R N) < \infty$, $r = 1$ and $e = 1$ we have:

$$\eta_1^R(M, N) = \frac{1}{2} \eta_0^{R'}(M, N)$$

Proof. Let $n > h$ be integers. Let $g_{M,N}^R(h, n) = \sum_{i=h}^n (-1)^i \beta_i^R(M, N)$. Then, for a fixed h , it is easy to see that:

$$\eta_e^R(M, N) = \lim_{n \rightarrow \infty} \frac{g_{M,N}^R(h, n)}{n^e}$$

(1) If $e = 0$ then $\text{lcx}_R(M, N) = 0$, so $\beta_i = 0$ for $i \gg 0$, and there is nothing to prove. Assume $e > 0$. We choose an integer h is big enough so that the formula for $\beta_i(M, N)$ in 4.1 is true for all $i \geq h$. Hence :

$$\begin{aligned}
g_{M,N}^R(h, n) &= \sum_{i=h}^n (-1)^i \beta_i \\
&= \frac{m_0}{(c-1)!} \sum_h^n (-1)^i i^{c-1} + \frac{n_0}{(d-1)!} \sum_h^n i^{d-1} + \sum_h^n (-1)^i g_{(-1)^i}(i)
\end{aligned}$$

Since the first and third terms are of order n^{c-1} or less, and $e \geq d$, we have:

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{g_{M,N}^R(h, n)}{n^r} &= \lim_{n \rightarrow \infty} \frac{n_0}{(d-1)!} \frac{\sum_h^n i^{d-1}}{n^e} \\
&= \lim_{n \rightarrow \infty} \frac{n_0}{d!} n^{d-e}
\end{aligned}$$

Since $d \leq c \leq e$, the limit is finite. The second statement is also obvious.

(2) It is enough to prove the first equation, since the second one follows in an identical manner. The short exact sequence $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ gives rise to the long exact sequence :

$$\dots \rightarrow \text{Tor}_i(M_1, N) \rightarrow \text{Tor}_i(M_2, N) \rightarrow \text{Tor}_i(M_3, N) \rightarrow \text{Tor}_{i-1}(M_1, N) \rightarrow \dots$$

which we truncate at $n > h > \max\{f_R(M_1, N), f_R(M_2, N), f_R(M_3, N)\}$ to get :

$$\begin{aligned}
0 \rightarrow B_n \rightarrow \text{Tor}_n(M_1, N) \rightarrow \text{Tor}_n(M_2, N) \rightarrow \text{Tor}_n(M_3, N) \rightarrow \dots \rightarrow \text{Tor}_h(M_3, N) \\
\rightarrow C_h \rightarrow 0
\end{aligned}$$

$$g_{M_1, N}(h, n) - g_{M_2, N}(h, n) + g_{M_3, N}(h, n) = \pm \ell(B_n) \pm \ell(C_h)$$

Divide by n^e and take the limit as $n \rightarrow \infty$. Since h is fixed and $\ell(B_n) \leq \ell(\text{Tor}_n(M_1, N))$, which is of order $n^{\text{lcx}(M_1, N)-1}$, the right hand side must be 0. The left hand side gives exactly the equality we seek.

(3) We will make use of the long exact sequence for change of rings in 2.2:

$$\dots \rightarrow \text{Tor}_{i-1}^R(M, N) \rightarrow \text{Tor}_i^{R'}(M, N) \rightarrow \text{Tor}_i^R(M, N) \rightarrow \text{Tor}_{i-2}^R(M, N) \rightarrow \dots$$

Again, choose $n > h > f_R(M, N)$ and we truncate as follows :

$$\begin{aligned}
0 \rightarrow B_n \rightarrow \text{Tor}_{n+1}^R(M, N) \\
\rightarrow \text{Tor}_{n-1}^R(M, N) \rightarrow \text{Tor}_n^{R'}(M, N) \rightarrow \text{Tor}_n^R(M, N) \\
\rightarrow \dots \\
\rightarrow \text{Tor}_h^R(M, N) \rightarrow \text{Tor}_{h+1}^{R'}(M, N) \rightarrow \text{Tor}_{h+1}^R(M, N) \rightarrow C_h \rightarrow 0
\end{aligned}$$

Take the alternate sum of lengths to get:

$$g_{M,N}^{R'}(0, n) = (-1)^{n+1} \ell(\text{Tor}_{n+1}^R(M, N)) + (-1)^n \ell(\text{Tor}_n^R(M, N)) \pm \ell(B_n) \pm \ell(C_h)$$

Now we make use of Theorem 4.1, observing that since B_n is a quotient of $\text{Tor}_{n+1}^{R'}(M, N)$, $\ell(B_n)$ is of order n^{e-2} or less. Thus:

$$g_{M,N}^{R'}(h, n) = \frac{n_0}{(d-1)!} (n^{d-1} + (n+1)^{d-1}) + f(n)$$

where $f(n)$ is of order n^{e-2} . Divide by n^{e-1} and take limit as $n \rightarrow \infty$, we get :

$$\begin{aligned}\eta_{e-1}^{R'}(M, N) &= 2d \lim_{n \rightarrow \infty} \frac{n_0}{d!} n^{d-e} \\ &= 2e \lim_{n \rightarrow \infty} \frac{n_0}{d!} n^{d-e} \\ &= 2e \eta_e^R(M, N)\end{aligned}$$

The second equality follows because the limit is nonzero if and only if $d = e$. The third equality follows from the last line of part (1). The last statement of (3) can be proved in a similar (but simpler) manner, and in any case can be found in [Ho1]. \square

Remark 4.4. We want to note that $\eta_e^R(M, N)$ can be viewed as a natural extension of some familiar notions. When R is regular and $\ell(M \otimes_R N) < \infty$, $\eta_0^R(M, N) = \chi^R(M, N)$, Serre's intersection multiplicity. When $R = Q/(f)$ is a hypersurface, then $2\eta_1^R(M, N) = \theta^R(M, N)$, the function defined by Hochster in [Ho1] and studied in details in [Da1] and [Da2]. When R is a complete intersection of codimension r and $\ell(M \otimes_R N) < \infty$ then $\eta_r^R(M, N)$ agrees (up to a constant factor) with a notion defined by Gulliksen in [Gu]. We recall Gulliksen's definition. When R is a complete intersection of codimension r and $\ell(M \otimes_R N) < \infty$ then $\text{Tor}_*^R(M, N)$ is Artinian over $R[x_1, \dots, x_r]$ and we can write:

$$P_{M,N}^R(t) = \frac{p(t)}{(1-t^2)^r}$$

Then Gulliksen defined:

$$\chi^R(M, N) = p(-1)$$

Since almost Artinian modules behave less well than Artinian ones (see proof of 3.2), and in any case finer information are needed for our main results, we could not repeat Gulliksen's idea.

5. COMPARISON OF COMPLEXITIES

This section is devoted to the study of relationships between the different notions of complexities over complete intersections (which we defined in section 2). We again let (R, \mathfrak{m}, k) and (Q, \mathfrak{n}, k) be local rings such that $R = Q/(f_1, \dots, f_r)$, Q is regular and (f_1, \dots, f_r) is a regular sequence on Q . We denote by x_i , $i = 1, \dots, r$ the cohomology operators and let $S = R[x_1, \dots, x_r]$. Let M, N be R -modules.

Throughout this section, let $\mathcal{T} = \text{Tor}_*^R(M, N)$. The main result states that if $f_R(M, N) = f_R(\mathcal{T}) < \infty$ then:

$$\text{lc}_R(M, N) = \text{tc}_R(M, N) = \text{cx}_R(M, N)$$

We first note an easy:

Lemma 5.1. *Suppose there is an \mathfrak{m}_R -primary ideal I such that I kills $\mathcal{T}_{\geq n}$ for some integer n . Then $\text{lc}_R(M, N) = \text{tc}_R(M, N)$.*

Proof. Suppose i is an integer such that I kills $\text{Tor}_i^R(M, N)$. Let $\mu_i = \ell(\text{Tor}_i^R(M, N))$ and $a = \ell(R/I)$. Then by 2.10 : $a\mu_i \geq \beta_i \geq \mu_i$. Since this is true for all $i \gg 0$, we have $\text{cx}(\{\mu_i\}) = \text{cx}(\{\beta_i\})$, which is what we want. \square

Next, we recall some notations and results from Avramov and Buchweitz's paper [AB1]. Let \tilde{k} denotes an algebraic closure of k . A residual algebraic closure of Q is a flat extension of local ring $Q \subseteq \tilde{Q}$ such that $\mathfrak{n}_{\tilde{Q}}$ is the maximal ideal $\tilde{\mathfrak{n}}$ of \tilde{Q} and the induced map $k \rightarrow \tilde{Q}/\tilde{\mathfrak{n}}$ is the embedding $k \hookrightarrow \tilde{k}$. Such extension always exists (see [AB1], 2.1).

Let $\mathcal{E} = \text{Ext}_R^*(M, N) \otimes_R k$, a module over $A = S \otimes_R k = k[x_1, \dots, x_r]$. Then the support variety $V^*(Q, R, M, N)$ was defined as the zero set (in \tilde{k}^r) of the annihilator of \mathcal{E} in A , plus 0. The key result here shows why this is an important object:

Theorem 5.2. ([AB1], 2.4, 2.5] *Let Q, R, M, N as above. Then :*

- (1) $\text{cx}^R(M, N) = \dim V^*(Q, R, M, N)$
- (2) *Let $0 \neq \bar{a} = (\bar{a}_1, \dots, \bar{a}_r) \in \tilde{k}^r$. Then $\bar{a} \in V^*(Q, R, M, N)$ if and only if $\text{Ext}_{Q_a}^n(\tilde{M}, \tilde{N}) \neq 0$ for infinitely many n . Here $\tilde{M} = M \otimes_R \tilde{Q}$, $a = (a_1, \dots, a_r)$ is a lift of \bar{a} in \tilde{Q}^r , $f_a = \sum a_i f_i$ and $Q_a = \tilde{Q}/(f_a)$.*

Now, suppose $n = f_R(\mathcal{T}) < \infty$. Then \mathcal{T} is almost Artinian, hence $\mathcal{T}_{\geq n}$ is an Artinian module over S . Let $\mathcal{T}_{\geq n}^\vee$ denote $\bigoplus_{i \geq n} \text{Hom}_R(\mathcal{T}_i, E_R(k)) = \text{Hom}_{\text{graded } R\text{-modules}}(\mathcal{T}, E_R(k))$. Then $\mathcal{T}_{\geq n}^\vee$ becomes a graded module over $S' = R[x_1^\vee, \dots, x_r^\vee]$. Let $\mathcal{D} = \mathcal{T}_{\geq n}^\vee \otimes_R k$, a module over $A' = k[x_1^\vee, \dots, x_r^\vee]$. Since $\mathcal{T}_{\geq n}$ is Artinian over S , \mathcal{D} is Noetherian over A' by 2.11.

From the discussion above we can define a set :

$$V_*(Q, R, M, N) = Z(\text{ann}_{A'} \mathcal{D}) \cup \{0\} \subseteq \tilde{k}^r.$$

The next result is a dual version of theorem 5.2:

Lemma 5.3. *Let Q, R, M, N be as above such that $f_R(\mathcal{T}) < \infty$ (i.e. $\ell(\text{Tor}_i^R(M, N)) < \infty$ for $i \gg 0$). Then :*

- (1) $\text{tcx}^R(M, N) = \dim V_*(Q, R, M, N)$.
- (2) *Let $0 \neq \bar{a} = (\bar{a}_1, \dots, \bar{a}_r) \in \tilde{k}^r$. Then $\bar{a} \in V_*(Q, R, M, N)$ if and only if $\text{Tor}_n^{Q_a}(\tilde{M}, \tilde{N}) \neq 0$ for infinitely many n . Here $\tilde{M} = M \otimes_R \tilde{Q}$, $a = (a_1, \dots, a_r)$ is a lift of \bar{a} in \tilde{Q}^r , $f_a = \sum a_i f_i$ and $Q_a = \tilde{Q}/(f_a)$.*

Proof. (1) Let $n = f_R(\mathcal{T})$, then $\mathcal{T}_{\geq n}$ is Artinian. Since $\mathcal{T}_{\geq n}^\vee$ is Noetherian, by 2.4 the annihilators of \mathcal{T}_i^\vee become periodic of period 2 eventually. So the same thing is true for the annihilators of \mathcal{T}_i . Lemma 5.1 shows that $\text{lcx}_R(M, N) = \text{tcx}_R(M, N)$. We now have:

$$\text{tcx}_R(M, N) = \text{lcx}_R(M, N) = \text{cx}_R(\mathcal{T}) = \text{cx}_R(\mathcal{T}_{\geq n}^\vee) = \text{cx}_R(\mathcal{D}) = \dim(V_*(Q, R, M, N))$$

(2) Without loss of generality, we may assume $\bar{a} = (1, 0, \dots, 0)$ and $Q = \tilde{Q}$. Then $f_a = f_1$. Let $Q_1 = Q/(f_1)$. Then $R = Q_1/(f_2, \dots, f_r)$ and so \mathcal{T} is a module over $S_1 = R[x_2, \dots, x_r]$ (the actions here agree with the actions from S). By Proposition 2.11, $\text{Tor}_*^{Q_1}(M, N)$ is almost Artinian over $Q_1 \iff \mathcal{T}$ is almost Artinian over $S \iff \mathcal{D}$ is a finite module over $A'_1 = k[x_2^\vee, \dots, x_r^\vee] \iff (1, 0, \dots, 0) \notin V_*(Q, R, M, N) \quad \square$

In summary:

Theorem 5.4. *Let R be a local complete intersection and M, N be R modules such that $f_R(\mathcal{T}) < \infty$. Then $\text{lcx}_R(M, N) = \text{tcx}_R(M, N) = \text{cx}_R(M, N)$.*

Proof. The first equality was proved in part (1) of Lemma 5.3. The second equality follows part (2) of 5.2, part (2) of 5.3 and the fact that over the hypersurface Q_a ,

$\mathrm{Tor}_n^{Q_a}(\widetilde{M}, \widetilde{N}) \neq 0$ for infinitely many n if and only if $\mathrm{Ext}_{Q_a}^n(\widetilde{M}, \widetilde{N}) \neq 0$ for infinitely many n (see [AB1], 5.12 and [HW1], 1.9). \square

As a corollary we reprove a result by Avramov and Buchweitz, which says that the vanishing of all higher Tor modules is equivalent to the vanishing of all higher Ext modules.

Corollary 5.5. ([AB1], 6.1) *Let R be a local complete intersection and M, N be R -modules. Then $\mathrm{Ext}_R^i(M, N) = 0$ for $i \gg 0$ if and only if $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$.*

Proof. The statement is equivalent to saying that $\mathrm{cx}_R(M, N) = 0$ if and only if $\mathrm{tcx}_R(M, N) = 0$. We will use induction on $d = \dim R$. If $\dim R = 0$ then all the Tor modules have finite length, so we can apply 5.4 directly. If $d > 0$, and $\mathrm{Tor}_i^R(M, N) = 0$ for $i \gg 0$, then 5.4 also applies. Assume that $\mathrm{Ext}_R^i(M, N) = 0$ for $i \gg 0$. By localizing at prime ideals in the punctured spectrum of R and using induction hypothesis, we get that all the high Tor modules have finite length, so we can apply 5.4 again. \square

Remark 5.6. It is not known whether $\mathrm{cx}_R(M, N) = \mathrm{tcx}_R(M, N)$ when R is a local complete intersection and (M, N) is any pair of R -modules. Avramov and Buchweitz's result says that $\mathrm{cx}_R(M, N) = 0$ if and only if $\mathrm{tcx}_R(M, N) = 0$. Our result 5.4 shows that $\mathrm{cx}_R(M, N) = \mathrm{tcx}_R(M, N)$ when $f_R(M, N) < \infty$.

Corollary 5.7. *Let R be a local complete intersection and M, N be R -modules such that $f_R(M, N) < \infty$. Then $\mathrm{tcx}_R(M, N) \leq \min\{\mathrm{cx}_R M, \mathrm{cx}_R N\}$.*

Proof. This follows from theorem 5.4 and the fact that $\mathrm{cx}_R(M, N) \leq \min\{\mathrm{cx}_R M, \mathrm{cx}_R N\}$ (corollary 5.7, [AB1]). \square

6. C-RIGIDITY

We will say that (M, N) is *c-rigid* if the vanishing of c consecutive Tor's forces the vanishing of all higher Tor's. The module M is called *c-rigid* if (M, N) is *c-rigid* for all finitely generated N . When $c = 1$ we simply say that (M, N) (or M) is *rigid*. We first recall some notations and results from [Ho1] and [Da1]:

A local ring (R, m, k) is a *admissible complete intersection* if $\hat{R} \cong Q/(f_1, \dots, f_r)$, f_1, \dots, f_r form a regular sequence on Q and Q is a power series ring over a field or a discrete valuation ring.

The function $\theta^R(M, N)$

Let $R = T/(f)$ be an admissible local hypersurface. The function $\theta^R(M, N)$ was introduced by Hochster ([Ho1]) for any pair of finitely generated modules M, N such that $f_R(M, N) < \infty$ as:

$$\theta^R(M, N) = \ell(\mathrm{Tor}_{2e+2}^R(M, N)) - \ell(\mathrm{Tor}_{2e+1}^R(M, N)).$$

where e is any integer $\geq (d+2)/2$. It is well known (see [Ei]) that $\mathrm{Tor}^R(M, N)$ is periodic of period 2 after $d+1$ spots, so this function is well-defined. The theta function satisfies the following properties. First, if $M \otimes_R N$ has finite length, then:

$$\theta^R(M, N) = \chi^T(M, N).$$

Secondly, $\theta^R(M, N)$ is biadditive on short exact sequence, assuming it is defined. Specifically, for any short exact sequence:

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$$

and any module M such that $f_R(M, N_i) < \infty$ for all $i = 1, 2, 3$, we have $\theta^R(M, N_2) = \theta^R(M, N_1) + \theta^R(M, N_3)$. Similarly, $\theta(M, N)$ is additive on the first variable.

In [Da1], we show that when $\theta^R(M, N)$ can be defined and vanishes, then (M, N) is rigid:

Proposition 6.1. *Let R be an admissible hypersurface and M, N be R -modules such that $f_R(M, N) < \infty$ (so that $\theta^R(M, N)$ can be defined). Assume $\theta^R(M, N) = 0$. Then (M, N) is rigid.*

By Remark 4.4, $\theta^R(M, N) = 2\eta_1^R(M, N)$ (when both are defined). Using our generalized function $\eta^R(M, N)$, it is easy to get similar results on c-rigidity over complete intersections. We first isolate a simple corollary of 2.2, whose proof we will omit:

Corollary 6.2. *Let Q be a Noetherian ring with f a nonzerodivisor on Q . Let $R = Q/(f)$ and M, N be R -modules. Let n, i, c be integers.*

(1) *If $\text{Tor}_Q^i(M, N) = 0$ for all $i \geq n$ then $\text{Tor}_i^R(M, N) \cong \text{Tor}_{i+2}^R(M, N)$ for all $i \geq n-1$.*

(2) *If $\text{Tor}_R^i(M, N) = 0$ for $n \leq i \leq n+c$ then $\text{Tor}_Q^i(M, N) = 0$ for $n+1 \leq i \leq n+c$.*

Theorem 6.3. *Let R be a codimension $r > 0$ admissible complete intersection and M, N be R -modules. Assume that $f_R(M, N) < \infty$ and $\eta_r^R(M, N) = 0$. Then (M, N) is r -rigid.*

Proof. We use induction on r . The case $r = 1$ is Proposition 6.1. We may assume $R = Q/(f_1, \dots, f_r)$ where Q is regular. Suppose $\text{Tor}_R^i(M, N) = 0$ for $n+1 \leq i \leq n+r$ for some integer n . Let $R' = Q/(f_2, \dots, f_r)$. Then by Corollary 6.2 we have $\text{Tor}_{R'}^i(M, N) = 0$ for $n+2 \leq i \leq n+r$. By Theorem 4.3 $\eta_{r-1}^{R'}(M, N) = 0$ so by induction hypothesis (M, N) is $(r-1)$ -rigid as R' module. Thus $\text{Tor}_{R'}^i(M, N) = 0$ for $i \geq n+2$ and by 6.2 again we have $\text{Tor}_R^i(M, N) = 0$ for $i \geq n+1$. \square

Corollary 6.4. *Let R be a codimension r admissible complete intersection and M, N be R -modules. Assume the $\text{pd}_{R_p} M_p < \infty$ for all $p \in Y(R)$ and $[N] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$, the reduced Grothendieck group finitely generated modules over R with rational coefficients. Then (M, N) is r -rigid.*

Proof. The proof is identical to the hypersurface case (see [Da1], 4.3) using 6.3 instead of 6.1. \square

Corollary 6.5. *Let R be an codimension $r > 0$ admissible complete intersection and M be an R -module such that $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. Let $\text{IPD}(M) := \{p \in \text{Spec}(R) \mid \text{pd}_{R_p} M_p = \infty\}$. Assume that $\text{IPD}(M)$ is either \emptyset or is equal to $\text{Sing}(R)$. Then M is r -rigid.*

Proof. The proof is identical to the hypersurface case (see [Da1], 4.5) using 6.3 instead of 6.1. \square

Corollary 6.6. *Let R be a codimension $r > 0$ admissible complete intersection and M, N be R -modules. Assume:*

(1) $M \otimes_R N$ has finite length.

(2) $\dim(M) + \dim(N) \leq \dim(R) + r - 1$.

Then (M, N) is r -rigid. In particular, a finite length module is r -rigid

Proof. The case $r = 1$ is 4.4 in [Da1]. Then 6.2 allows us to use induction. \square

Remark 6.7. We want to point out that instead of using $\eta_c^R(M, N)$, one can appeal to $\theta^{R_1}(M, N)$ (here $R_1 = Q/(f_1)$) to prove some of the above results. We will sketch a proof for 6.4. The point is that the hypotheses on M and N lift to R_1 . So M, N are rigid over R_1 . Now using the change of rings exact sequence repeatedly shows that M, N are c -rigid over R .

We also note this generalization of a result by Lichtenbaum in [Li]) which says that over an admissible hypersurface, a module of finite projective dimension is rigid:

Corollary 6.8. *Let R be a codimension $r > 0$ admissible complete intersection, and M be an R -module such that $\text{cx}(M) \leq r - 1$. Then M is r -rigid.*

Proof. Let N be any R -module we want to show that (M, N) is r -rigid. We use induction on $\dim R$. If $\dim R = 0$, all modules have finite length, so by 5.7 we have $\text{tex}(M, N) \leq \text{cx}(M)$. Thus $\eta_r^R(M, N) = 0$ and (M, N) is r -rigid by Theorem 6.3. Assume $\dim R > 0$. By induction (through localizing at all primes $p \in Y(R)$) we have $f_R(M, N) < \infty$ so we can apply 5.7 and 6.3 again to show that $\eta_r^R(M, N) = 0$. \square

In general, when a powerful rigidity result is not present, we have to be content with “being rigid after certain point”. We will say that (M, N) is (c, n) -rigid if $\text{Tor}_i^R(M, N) = 0$ for $N \leq i \leq N + c - 1$ with $N > n$ forces the vanishing of all $\text{Tor}_i^R(M, N)$ for $i \geq N$. The module M is called (c, n) -rigid if (M, N) is (c, n) -rigid for all finitely generated N .

Lemma 6.9. *Let Q be a local ring with f a nonzerodivisor on Q . Let $R = Q/(f)$ and M, N be R -modules such that $f_R(M, N) < \infty$ and $\text{pd}_Q M < \infty$ (so $\eta_1^R(M, N)$ can be defined by 4.3). If $\eta_1^R(M, N) = 0$ then (M, N) is $(1, \text{depth } R - \text{depth } M)$ -rigid.*

Proof. Since $\text{pd}_Q M < \infty$ we have $\text{Tor}_i^R(M, N) = 0$ for $i > \text{depth } Q - \text{depth } M$. By 6.2 we have $\ell(\text{Tor}_i^R(M, N)) = \ell(\text{Tor}_{i+2}^R(M, N))$ for all $i > \text{depth } Q - \text{depth } M - 1 = \text{depth } R - \text{depth } M$. But then the condition $\eta_1^R(M, N) = 0$ forces $\ell(\text{Tor}_i^R(M, N)) = \ell(\text{Tor}_{i+1}^R(M, N))$ for all $i > \text{depth } R - \text{depth } M$ and the conclusion follows trivially. \square

Theorem 6.10. *Let R, Q be local rings such that $R = Q/(f_1, \dots, f_r)$ and f_1, \dots, f_r is a regular sequence on Q . Let M, N be R modules such that $f_R(M, N) < \infty$ and $\text{pd}_Q M < \infty$ (so $\eta_r^R(M, N)$ can be defined by 4.3). If $\eta_r^R(M, N) = 0$ then (M, N) is $(r, \text{depth } R - \text{depth } M)$ -rigid.*

Proof. The proof goes by induction using 6.9 as the base case and 4.3 for the inductive step similarly to the proof of 6.3. \square

7. APPLICATIONS TO TENSOR PRODUCTS OVER COMPLETE INTERSECTIONS

In this section we aim to prove results that can be viewed as generalizations of Auslander's classical theorem that, over a regular local ring R , $M \otimes_R N$ is torsion-free implies that $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$. This conclusion forces M, N to satisfy the “depth formula” :

$$\text{depth}(M) + \text{depth}(N) = \text{depth}(R) + \text{depth}(M \otimes_R N).$$

An attempt to extend Auslander's result for complete intersections was made in [HW1] and [HJW]. We first collect some notation and results. Recall that $X^i(R)$ denotes the set of prime ideals p of height less than or equal to i in R (since we only consider complete intersections, this also means $\text{depth}(R_p) \leq i$).

The Condition (S_n)

For a non-negative integer n , M is said to satisfy (S_n) if:

$$\text{depth}_{R_p} M_p \geq \min\{n, \dim(R_p)\} \quad \forall p \in \text{Spec}(R)$$

(The depth of the 0 module is set to be ∞). This definition was taken from “Syzygies” (see [page3, EG]).

The Pushforward

Let R be a Gorenstein ring and M a torsion-free (equivalent to (S_1)) R -module. Consider a short exact sequence :

$$0 \rightarrow W \rightarrow R^\lambda \rightarrow M^* \rightarrow 0$$

Here λ is the minimal number of generators for M^* . Dualizing this short exact sequence and noting that M embeds into M^{**} we get an exact sequence:

$$0 \rightarrow M \rightarrow R^\lambda \rightarrow M_1 \rightarrow 0$$

This exact sequence is called the *pushforward* of M .

We record a result on pushforward in [HJW] below for the reader's convenience. Note that since their definition of (S_n) contains some inconsistency with the literature, some minor details need to be fixed. See [HW3] for details, and also proof of 7.2 below.

Proposition 7.1. ([HJW], 1.6) *Let R, M, M_1 as above. Then for any $p \in \text{Spec}(R)$:*

- (1) M_p is free if and only if $(M_1)_p$ is free.
- (2) If M_p is a maximal Cohen-Macaulay R_p -module, then so is $(M_1)_p$.
- (3) $\text{depth}_{R_p}(M_1)_p \geq \text{depth}_{R_p} M_p - 1$.
- (4) If M satisfies (S_k) , then M_1 satisfies (S_{k-1}) .

Proposition 7.2. ([HJW], 2.1) *Let R be a Gorenstein ring, let $c \geq 1$ and let M, N be R -modules such that:*

- (1) M satisfies (S_c) .
 - (2) N satisfies (S_{c-1}) .
 - (3) $M \otimes_R N$ satisfies (S_c) .
 - (4) M_p is free for each $p \in X^{c-1}(R)$.
- Put $M_0 := M$ and, for $i = 1, \dots, c$, let

$$0 \rightarrow M_{i-1} \rightarrow F_i \rightarrow M_i \rightarrow 0$$

be the pushforward. Then $\text{Tor}_i^R(M_c, N) = 0$ for $i = 1, \dots, c$.

Proof. In view of [HW3], the proof in [HJW] only needs one more line. In the inductive step, instead of looking at any $p \in \text{Spec}(R)$, we localize at $p \in \text{Supp}(M_i \otimes_R N)$ (which means p is also in $\text{Supp}(M_i), \text{Supp}(N)$). \square

For the reader's conveniences we collect below what have been done to extend Auslander's theorem mentioned at the beginning of this section.

Theorem 7.3. ([HW1], 2.7) *Let R be a hypersurface and M, N be finitely generated R -modules, one of which has constant rank (on the associated primes of R). If $M \otimes_R N$ is reflexive, then $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$.*

Theorem 7.4. ([HJW], 2.4). *Let R be a codimension 2 complete intersection and M, N be finitely generated R -modules. Assume:*

- (1) M is free of constant rank on $X^1(R)$.
- (2) N is free of constant rank on $X^0(R)$.
- (3) M, N satisfy (S_2) .

If $M \otimes_R N$ satisfies (S_3) , then $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$.

Theorem 7.5. ([HJW], 2.8). *Let R be a codimension 3 admissible complete intersection and M, N be finitely generated R -modules. Assume:*

- (1) M is free of constant rank on $X^2(R)$.
- (2) N is free of constant rank r on $X^1(R)$ and $(\wedge^r N)^{**} \cong R$. (Such a module N is called orientable).
- (3) M, N satisfy (S_3) .

If $M \otimes_R N$ satisfies (S_4) , then $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$.

Our aim is to prove the following result:

Theorem 7.6. *Let c be any integer greater or equal to 1. Let R be a codimension c admissible complete intersection and M, N be finitely generated R -modules. Assume:*

- (1) M is free on $X^c(R)$.
- (2) M, N satisfy (S_c) .

If $M \otimes_R N$ satisfies (S_{c+1}) , then $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$.

Our strategy will be to use the c -rigidity results in the previous section. The next lemma is critical for our proof:

Lemma 7.7. *Let R be an admissible complete intersection of codimension $c > 0$ and M, N be R -modules. Assume that:*

- (1) $\text{Tor}_i^R(M, N) = 0$ for $1 \leq i \leq c$.
- (2) $\text{depth}(N) \geq 1$ and $\text{depth}(M \otimes_R N) \geq 1$.
- (3) $f_R(M, N) < \infty$.

Then $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$.

Proof. The depth assumptions ensure that we can choose t a nonzero divisor for both N and $M \otimes_R N$. Let $\bar{N} = N/tN$. Tensoring the short exact sequence :

$$0 \longrightarrow N \xrightarrow{t} N \longrightarrow \bar{N} \longrightarrow 0$$

with M we get :

$$0 \longrightarrow \text{Tor}_1^R(M, \bar{N}) \longrightarrow M \otimes_R N \xrightarrow{t} M \otimes_R N \longrightarrow M \otimes_R \bar{N} \longrightarrow 0$$

which shows that $\text{Tor}_1^R(M, \bar{N}) = 0$. Together with condition (1), this shows $\text{Tor}_i^R(M, \bar{N}) = 0$ for $1 \leq i \leq c$. But condition (3) is satisfied for both pair M, N and M, \bar{N} so :

$$\eta_c^R(M, \bar{N}) = \eta_c^R(M, N) - \eta_c^R(M, N) = 0$$

The conclusion then follows from theorem 6.3 and Nakayama's lemma. \square

Proof. (of theorem 7.6). We use induction on $d = \dim(R)$. For $d \leq c$, condition (1) forces M to be free. Suppose $d = c + 1$. Following proposition 7.2 we have a sequence of modules $M_0 = M, M_1, \dots, M_c$ such that there are exact sequences:

$$0 \rightarrow M_{i-1} \rightarrow F_i \rightarrow M_i \rightarrow 0$$

By the conclusion of 7.2, for each i , $\text{Tor}_j^R(M_i, N) = 0$ for $1 \leq j \leq i$. So we have a short exact sequence :

$$0 \rightarrow M_{i-1} \otimes_R N \rightarrow F_i \otimes_R N \rightarrow M_i \otimes_R N \rightarrow 0$$

By assumption (2), (3) and the fact that $d = c + 1$ we have $\text{depth}(M \otimes_R N) \geq c + 1$ and $\text{depth}(N) \geq c$. Using the “depth lemma” repeatedly, we can conclude that $\text{depth}(M_c \otimes_R N) \geq 1$. Now, condition (1) means that M is free on the punctured spectrum of R , and so $f_R(M_c, N) < \infty$. So lemma 7.7 applies.

Now assume $d > c + 1$. By the induction hypothesis, $\text{Tor}_i^R(M, N)$ has finite length for $i \geq 1$. So an identical argument to the one above together with lemma 7.7 give the desired conclusion. \square

8. SOME APPLICATIONS ON INTERSECTION THEORY OVER COMPLETE INTERSECTIONS

We present here a few other examples where one can exploit the properties of the function η^R proved in the last chapter. The first one extends Hochster's result on dimensional inequality ([Ho1]).

Theorem 8.1. *Let $R = Q/(f_1, \dots, f_r)$ be a local complete intersection with Q a regular local ring satisfying Serre's Positivity conjecture (if $r = 0$, then $R = Q$). Let M be an R -module such that $\text{Supp}(M)$ contains the singular locus $\text{Sing}(R)$ of R and $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$. Then for any R module N such that $M \otimes_R N$ is finite length we have: $\dim M + \dim N < \dim R + r$.*

Proof. Since $\text{Supp}(M) \cap \text{Supp}(N) = \{m\}$ we have $\text{Sing}(R) \cap \text{Supp}(N) \subset \{m\}$. So $\text{IPD}(N) \subset \{m\}$. Therefore for any R -module N' , $f_R(N', N) < \infty$ and $\eta_r^R(N', N)$ is defined. Then the fact that $[M] = 0$ in $\overline{G}(R)_{\mathbb{Q}}$ and the biadditivity of η (part (2), 4.3) force $\eta_r^R(M, N) = 0$. Then part (3) of 4.3 indicates that $\chi^Q(M, N) = 0$ and we must have $\dim M + \dim N < \dim Q = \dim R + r$. \square

The following example shows that the inequality above is sharp:

Example 8.2. Let k be an algebraically closed field of characteristic 0. Let $Q = k[x_{ij}], 0 \leq i \leq r, 1 \leq j \leq r$ and $m = (x_{ij})$. For $1 \leq i \leq r$, let $y_i = \sum_{j=1}^r x_{0j}x_{ij}$. Let $R = Q_m/(y_1, \dots, y_r)$. Then R is a local complete intersection of codimension r and $\dim R = r^2$.

Let A be the square matrix $[x_{ij}]_{1 \leq i \leq r, 1 \leq j \leq r}$ and $d = \det(A)$. Let $I = (x_{01}, \dots, x_{0r}, d)$ and J be the ideal generated by the entries of A . Then

$$\dim R/I + \dim R/J = (r^2 - 1) + r = \dim R + r - 1$$

The short exact sequence :

$$0 \longrightarrow R/(x_{01}, \dots, x_{0r}) \xrightarrow{d} R/(x_{01}, \dots, x_{0r}) \longrightarrow R/I \longrightarrow 0$$

shows that R/I is 0 in the Grothendieck group of R . It remains to show $V(I)$ contains the singular locus of R . But $\text{Sing}(R) = V(I')$, where I' is generated by the r -by- r determinants of $B = \left[\frac{\partial(y_i)}{\partial x_{ij}} \right]$. It is easy to see that I' contains $d, x_{01}^r, \dots, x_{0r}^r$, therefore $V(I) \supseteq V(I')$ (they are actually equal).

The next corollary gives an asymptotic version of the Vanishing Theorem for complete intersections due to Roberts (cf. [Ro], 13.1.1):

Corollary 8.3. *Let $R = Q/(f_1, \dots, f_r)$ be a local complete intersection with Q a regular local ring and M, N be finitely generated R -modules such that $M \otimes_R N$ has finite length. Let $a = \max\{\text{cx}_R(M), \text{cx}_R(N)\}$. Suppose that $\dim M + \dim N < \dim R + a$. Then $\eta_a^R(M, N) = 0$.*

Proof. Let $b = r - a$. By theorem 9.3.1 in [Av2] we can factor the surjection $Q \rightarrow R$ as $Q \rightarrow R' \rightarrow R$ such that the kernels of both maps are generated by regular sequence, with the first one having length b , and $\text{cx}_{R'}(M) = \text{cx}_{R'}(N) = 0$. Applying Roberts' theorem and 4.3, we see that the result follows. \square

In our view, the greatest potential of this study is the link between asymptotic homological algebra, a new and rapidly developing field, and the classical homological questions. To further illustrate this link, let us look at a well-known unsolved question (see [PS]), inspired by Serre's results on intersection multiplicity:

Question 8.4. Let R be a local ring and M, N be R -modules such that $\ell(M \otimes_R N) < \infty$ and $\text{pd}_R M < \infty$. Is it always true that $\dim M + \dim N \leq \dim R$?

In view of 8.3, we would like to pose the following “asymptotic” form of the above question:

Question 8.5. Let R be a local complete intersection and M, N be R -modules such that $\ell(M \otimes_R N) < \infty$. Is it always true that:

- (1) $\dim M + \dim N \leq \dim R + \text{cx}(M)$?
- (2) $\dim M + \dim N \leq \dim R + \text{tcx}(M, N)$?

Obviously, in view of 5.7, part (2) of the above (if true) would be stronger than part (1). Also, part (1) and (2) are equivalent for hypersurfaces and hold true if \hat{R} is a hypersurface in a unramified or equicharacteristic regular local ring (cf. [HW1], 1.9 and [Da1], 2.5).

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